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Resolution of the Transport equation subject to constraint

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Abstract

We present a method for solving the Transport equation when its solution has to belong to a constrained set which is not required to be convex. An autonomous formulation of the characteristics method allows us to use the tangency condition which has been introduced for ordinary differential equations. Thus we obtain a sufficient condition for existence of solutions, which shows the interplay between the geometry of the constraints set K and the velocity field β. A numerical method is proposed for solving the problem when the sufficient condition is not satisfied. A numerical experiment is presented showing the efficiency of the algorithm proposed.

MSC: 35F40; 35L85; 35R05

Keywords: Characteristic curves; Autonomous system; Set of non-convex constraints; Tangency condition

1. Introduction

Let u be a solution to the following Transport equation subject to constraints:

$$\begin{cases} (\beta|\nabla u) = f & \text{for } (t,x) \in Q, \\ u(t,x) = 0 & \text{for } (t,x) \in \partial Q_-, \\ u(t,x) \in K & \forall\, (t,x) \in Q, \end{cases} \tag{1}$$

where $Q = (0,T) \times \Omega \subset \mathbb{R}^{p+1}$ is the time–space domain,

$$\beta(t,x) = (1, \beta_2(t,x), \dots, \beta_{p+1}(t,x))^t$$

is the velocity field, $(\cdot,\cdot)$ stands for the Euclidean inner product of $\mathbb{R}^{p+1}$, and ∇ denotes the time–space gradient,

$$Tu = (\beta|\nabla u) = \partial_t u + \sum_{i=2}^{i=p+1} \beta_i \partial_{x_i} u.$$

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K is the constraints set. The Transport equation subject to constraints appears in many mathematical models. Let us quote for example reacting flows with convection [8], or in an image registration context the optical flow method [11]. When a least square formulation of the Transport equation is used, it has been proved in [16] that for a convex subset of constraints K the problem (1) has a unique solution. The aim of this article is to provide a time–space formulation of problem (1) where time and space variables have the same role in order to have existence and uniqueness results for this problem. Extending the so-called tangency condition known for differential equations subject to constraints (see for example, among a huge literature, [6,1,2,12]), a sufficient condition (15) is derived for problem (1) to have a unique solution. The interplay between the geometry of K, the velocity field β and the right-hand side f is exhibited, showing that the following simple strategy: solving then projecting, is not relevant for the issue under consideration. Then an efficient numerical algorithm is proposed to solve problem (1) when condition (15) is not satisfied.

The paper is organized as follows. The introduction is ended by recalling some standard results concerning the Lipschitzian domains according to [13]. The second section is dedicated to the formulation of the characteristic method in the form of an autonomous system. Some results established in [3] for velocities independent of time are extended to the time-dependent velocity cases. In the third section, the existence and uniqueness of solutions to problem (1) in the Banach space of the L^2 graph norm of the Transport operator is established. Finally in Section 4 an algorithm is proposed for solving problem (1) when the necessary condition is not satisfied. The algorithm is proved to be convergent, provided some monotonicity hypotheses are assumed to hold. Some numerical results are presented proving the efficiency of the proposed method.

The following results are classical, the reader is referred to [13,9] for a complete presentation. Here we introduce an isometry $\mathscr{T}_a$, which is implicitly defined with the Lipschitzian parametrization h in Necas:

(H0) Ω is a Lipschitzian bounded domain of $\mathbb{R}^p$.

Set $Q=(0,T)\times\Omega$ a time–space cylindrical bounded domain of $\mathbb{R}^{p+1}$. For any point a in ∂Q, there exist α_a a positive real, h_a a Lipschitzian application defined in $V_a=]-\alpha_a,\alpha_a[^p$ by

$$\begin{aligned} &h_a : V_a \longrightarrow \mathbb{R},\\ &z=(z_1,z_2,\ldots,z_p) \;\mapsto\; z_{p+1}=h_a(z_1,z_2,\ldots,z_p) \end{aligned}$$

a positive real δ_a and an isometry $\mathscr{T}_a$ from $\mathbb{R}^{p+1}$ onto $\mathbb{R}^{p+1}$, the associated vectorial isometry of which is denoted $\overrightarrow{\mathscr{T}}_a$, and which verifies:

(i) $\mathscr{T}_a\{(z,h_a(z))/z\in V_a\}\subset\partial Q$;
(ii) $\mathscr{T}_a\{(z,z_{p+1})/z\in V_a$ and $h_a(z)<z_{p+1}<h_a(z)+\delta_a\}\subset Q$;
(iii) $\mathscr{T}_a\{(z,z_{p+1})/z\in V_a$ and $h_a(z)-\delta_a<z_{p+1}<h_a(z)\}\subset\complement\overline{Q}$.

For every partition of unity $\{\psi_i\}_{i=1}^{i=m}$ associated to a covering of ∂Q with the open subsets

$$U_{a_i}=\{\mathscr{T}_{a_i}(z,h_{a_i}(z))/z\in V_{a_i},\ -\delta_{a_i}<z_{p+1}-h_{a_i}(z)<\delta_{a_i}\}\cap\partial Q \tag{2}$$

we introduce the carried surface measure (with Lipschitzian functions) on ∂Q and for $k\in\mathbb{N}$ we define the Banach space $L^k(\partial Q)$ by: $\varphi\in L^k(\partial Q)$ if φ is measurable and if

$$\int_{\partial Q}|\varphi|^k\,\mathrm{d}\sigma=\sum_{i=1}^{i=m}\int_{V a_i}(|\varphi|^k\psi_i)(z,h_{a_i}(z))\sqrt{1+\sum_{j=1}^{j=p}(D_j h_{a_i})^2(z)\,\mathrm{d}z}<\infty.$$

The Banach spaces so defined are independent of the partition of unity used [13, Theorem 4.1, p. 82, Lemmas 1.1, 1.2, p. 119 and p. 120].

1.1. Functional setting

We end this introduction with the functional space in which the solution will be sought. Let $\mathscr{D}(\overline{Q})$ denote the space of restrictions to $\overline{Q}$ of C^∞ functions with compact support in $\mathbb{R}^{p+1}$, and consider the norm

$$\|\varphi\|_{H(\beta,Q)} = \left(\|\varphi\|^2_{L^2(Q)} + \|\mathrm{div}(\beta\varphi)\|^2_{L^2(Q)} + \int_{\partial Q_-} |(\beta|n)|\varphi^2\,\mathrm{d}\sigma\right)^{1/2}.$$

We define the space $H_0(u, Q)$ as the closure of the subset $\{\varphi \in \mathscr{D}(\overline{Q}), \varphi|_{\partial Q_-} = 0\}$ for this norm:

$$H_0(\beta, Q) = \overline{\{\varphi \in \mathscr{D}(\overline{Q}), \varphi|_{\partial Q_-} = 0\}}^{H(\beta,Q)}.$$

If β is regular enough ($\beta \in L^2(0, T; H^1_0(\Omega))$) it can be seen that (see [5])

$$H_0(\beta, Q) \cap L^\infty(Q) = \{\rho \in L^2(Q), (\beta|\nabla\rho) \in L^2(Q), \rho|_{\partial Q_-} = 0 \text{ in } L^2(\partial Q_-, |(\beta|n)|\,\mathrm{d}\sigma)\} \cap L^\infty(Q).$$

Let us recall the following result (see [5]).

Theorem 1.1. *If* $\mathrm{div}(\beta) = 0$ *then the semi-norm* $|\cdot|_{1,\beta}$ *defined by*

$$|\varphi|_{1,\beta} = \|(\beta|\nabla\varphi)\|_{L^2(Q)}$$

is an equivalent norm in $H_0(\beta, Q)$.

2. Autonomous formulation for the characteristic curves

Concerning the velocity field we assume the following hypotheses to be satisfied:

(H1) There exists a bounded open set $Q_1 \subset \mathbb{R}^{p+1}$ verifying $\overline{Q} \subset Q_1$ and such that $\beta \in C^1(\overline{Q_1}; \mathbb{R}^{p+1})$ with $\beta = (1, 0, \dots, 0)^t$ in $\complement Q_1$; $f \in C^0(Q; \mathbb{R})$ and bounded.
(H2) $\mathrm{div}(\beta) = 0$.

The issue of this section is to prove that for almost every point $y \in Q$ there exists a characteristic curve issued from a point of ∂Q_- and containing y. The cornerstone for proving this result is Sard's theorem [3]. Since the velocity field depends on time, the ordinary differential system defining the characteristic curves is expressed as a differential autonomous system in time–space.

Lemma 2.1. *Assume the hypothesis* (H1) *to be satisfied, then* $\forall y \in \mathbb{R}_+ \times \mathbb{R}^p$ *there exists a unique solution* $Y(s, y) \in C^1(\mathbb{R}_+ \times \mathbb{R}^{p+1})$ *to*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} Y(s, y) = \beta(Y(s, y)) \quad \forall t \in \mathbb{R}_+, \\ Y(y_1, y) = y. \end{cases} \tag{3}$$

Moreover we have $Y_1(s, y) = s$ *for all* $s \in \mathbb{R}_+$ *and the orbits of the differential system* (3) *constitute a partition of* $\mathbb{R}^{p+1}$. *The application*

$$\begin{cases} \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}, \\ y \mapsto Y(s, y) \end{cases} \tag{4}$$

is a diffeomorphism of $\mathbb{R}^{p+1}$ *for all* $s \in \mathbb{R}_+$.

Proof. Theorem 1.2.2 of [10] applies thus the existence of the solution $Y(t, y)$ is proved in a neighborhood of y_1. Since $\overline{Q_1}$ is compact and $\beta \in C^1$, from hypothesis (H1) the right-hand side of system (3) can be bounded by a linear function of the unknown $Y(s, y)$. A consequence of Gronwall's lemma is the global existence with respect to time.

Trivially we have: $Y_1(s, y) - y_1 = s - y_1$. Since system (3) is autonomous, uniqueness implies that orbits do not intercept and constitute a partition of $\mathbb{R}^{p+1}$. $\square$

Later on we will need some notation concerning the characteristic curves. The characteristic curve containing the point y at time $t = y_1$ is denoted by

$$C_y = \{Y(s, y),\ s \in \mathbb{R}_+\}.$$

The incoming time $t^-(y)$ is defined with the connex component $\mathscr{I}_y$ of the set $\{s \in \mathbb{R}_+, Y(s, y) \in Q\}$ containing y_1:

$$\mathscr{I}_y = \{s \in [0, y_1] \text{ such that if } t \in [s, y_1] \text{ then } Y(t, y) \in Q\},$$

which is

$$t^-(y) = \inf(\mathscr{I}_y). \tag{5}$$

It is easy to see that $\mathscr{I}_y = (t^-(y), y_1]$. In a symmetric way, the outgoing time $t^+(y)$ is defined as to be the supremum of the interval

$$\{s \in [y_1, T] \text{ such that if } t \in [y_1, s] \text{ then } Y(t, y) \in Q\}.$$

We denote by $\tilde{C}_y$ the curve

$$\tilde{C}_y = \{Y(s, y),\ \ s \in [t^-(y), t^+(y)]\}.$$

Let ∂Q_0 denote the subset $\{a \in \partial Q \text{ s.t. } (N(a)|\beta = 0\}$, where $N(a)$ is the outward normal (which exists almost everywhere since the parametrization is locally Lipschitz). In the same way $\partial Q_\pm$ are defined by $\{a \in \partial Q \text{ s.t. } (N(a)|\beta) \gtrless 0\}$.

The set of irregular points of the boundary (i.e., the points where the parametrization is not differentiable) is denoted by ∂Q_i. Thus, the boundary is decomposed as follows:

$$\partial Q = \partial Q_i \cup \partial Q_0 \cup \partial Q_+ \cup \partial Q_-.$$

Now we present a technical lemma, the proof of which can be found in [14] (Lemma 3.1).

Lemma 2.2. *For all $y \in Q$, if $Y(t^-(y), y)$ is a regular point, then we have either*
$Y(t^-(y), y) \in \partial Q_-$ *or* $Y(t^-(y), y) \in \partial Q_0$.
For all $y \in Q$, if $Y(t^+(y), y)$ is a regular point, then we have either
$Y(t^+(y), y) \in \partial Q_+$ *or* $Y(t^+(y), y) \in \partial Q_0$.

Finally, this section is ended with a consequence of Sard's theorem.

Lemma 2.3. *Assume the hypotheses* (H0) *and* (H1) *to be satisfied, then*

- $\mathscr{M}^1$, *the union of characteristic curves issued from points in ∂Q_i (where the parametrization h is not a differentiable function), is a subset of Q of zero measure.*
- $\mathscr{M}^0$, *the union of characteristic curves issued from points in ∂Q_0, is a subset of Q of zero measure.*

Proof. Let us give a sketch of the proof. For a complete proof, the reader is referred to [14, Lemmas 3.1 and 3.4]. Let $\mathscr{M}_1$ be the subset of irregular points of ∂Q, $\mathscr{M}^1$ is the range of $[0, T] \times \mathscr{M}_1$ with the application χ defined by

$$\begin{aligned} \chi :\ & [0, T] \times \partial Q_- \to\ \mathbb{R}^{p+1}, \\ & (s, a)\ \mapsto Y(s, a). \end{aligned} \tag{6}$$

From (2), a partition of unity of ∂Q, we can argue locally, and we define $\gamma_{a_i}(z) = (z, h_{a_i}(z))$ for $z \in V_{a_i}$. The pre-domain of $U_{a_i} \cap \mathscr{M}_1$ through $\mathscr{T}_{a_i} \circ \gamma_{a_i}$ is $C_i = (\mathscr{T}_{a_i} \circ \gamma_{a_i})^{-1}(U_{a_i} \cap \mathscr{M}_1)$. Clearly C_i is the singular set points of the Lipschitzian application $\mathcal{T}_{a_i} \circ \gamma_{a_i}$, then the Rademacher theorem states that $C_i \subset \mathbb{R}^p$ has a zero measure. Introduce the application for a fixed $s \in [0, T]$, $z \mapsto g_s(z) = Y(s, \mathcal{T}_{a_i} \circ \gamma_{a_i}(z))$. This Lipschitzian application from $\mathbb{R}^p$ into $\mathbb{R}^{p+1}$ is such that $g_s(C_i)$ has a zero measure. Let s vary in $[0, T]$, we have the subset $\mathcal{M}^1$, all the first sections of which have zero measure, thus we deduce that the subset itself, belonging to a product of measured spaces, has a zero measure. That proves that the union of characteristic curves $\mathcal{M}^1$ has a zero measure.

The proof of the second part of the theorem is quite technical and uses Sard's theorem associated with the autonomous formulation of the characteristic curves, so we skip it (see [14, Lemmas 3.4]). $\square$

Thus, we have the following characteristics filling theorem [14, Theorem 3.3]:

Theorem 2.4. *Assume the hypotheses* (H0) *and* (H1) *to be satisfied, then there exists $\mathcal{M}$, a zero measure subset of Q, such that*

$$Q \backslash \mathcal{M} = \{Y(s, y), s \in]t^-(y), t^+(y)[, y \in \partial Q_-\}.$$

The next lemma will be useful for the definition at the solution to the Transport equation [14, Theorem 4.2].

Lemma 2.5. *Assume the hypotheses* (H0) *and* (H1) *to be satisfied, then there exists $\mathcal{M}$, a zero measure subset of Q such that the incoming time application $y \mapsto t^-(y)$ from $Q\backslash\mathcal{M}$ with values in $\mathbb{R}$ is continuous.*

Let us mention that a time–space formulation of the Transport equation is settled into a different way in [4] by introducing a measured space of $\mathbb{R}^{p+1}$ endowed with the product measures of ∂Q_- with the time, $\mathrm{d}\sigma \times \mathrm{d}s$, where s is the time needed to travel along the integral curve $\tilde{C}_y$, $y \in \partial Q_-$.

3. Existence of solutions belonging to the Banach space of the L^2 graph norm of the Transport operator

Let us define a solution to the Transport equation (1) by means of characteristic curves. Let $f \in C^0(Q)$ be given, then consider the function

$$\begin{aligned} &F : Q \times \mathbb{R} \to \mathbb{R}^{p+1} \times \mathbb{R}, \\ &(t, x, z) \mapsto (\beta(t, x), f(t, x)) \end{aligned} \tag{7}$$

and introduce the following autonomous ordinary differential system for the function $H = (Y, \varphi)^t$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} H = F(H), \\ \begin{pmatrix} Y(y_1, y) \\ \varphi(y_1) \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \end{cases} \quad y_1 < t, \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} Y(t, y) = \beta(Y(t, y)), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \varphi(t) = f(Y(t, y)), \\ Y(y_1, y) = y \text{ and } \varphi(y_1) = 0. \end{cases} \quad y_1 < t, \tag{8}$$

We have the following existence result:

Lemma 3.1. *Assume hypotheses* (H1) *and* (H2) *hold true. Then there exists a unique solution $H = (Y, \varphi)^t \in (C^1(Q) \times C^1(]t^-(y), t^+(y)[))$ to problem* (8).

Proof. Theorem 1.2.2 of [10] applies thus existence of the solution $Y(t, y)$ is proved in a neighborhood of y_1. Since from hypothesis (H1) the right-hand side of the first equation of system (8) is bounded by a linear function of the unknown $Y(t, y)$, a consequence of Gronwall's lemma is the global existence with respect to time. The existence and uniqueness of φ is straightforward since $f \in C^0$. $\square$

Now we are in position for introducing the definition of a solution to the Transport equation when $f \in C^0(Q)$. When $f \in L^2(Q)$, the same definition is still meaningful, provided a regularization procedure and a limit process are used as explained in the remark following the proof of Lemma 3.3.

Definition 3.2. For $f \in C^0(Q)$ and for $y \in Q \backslash \mathscr{M}$, the solution to the Transport equation (1) is defined by

$$\forall t \in]t^-(y), t^+(y)[, \quad u(y) = \int_{t^-(y)}^{y_1} f(Y(s, y))\, \mathrm{d}s = \varphi(t). \tag{9}$$

Let us prove that the function u belongs to the Banach space of the L^2 graph norm of the Transport operator.

Lemma 3.3. *Assume the hypotheses* (H1) *and* (H2) *hold true, then the function u defined by* (9) *verifies*

$$\|u\|_{L^2(Q)} \leqslant T \|f\|_{L^2(Q)}, \quad \|(\beta|\nabla u)\|_{L^2(Q)} \leqslant \|f\|_{L^2(Q)} \tag{10}$$

and is a weak solution in $H_0(\beta, Q)$ *to the Transport equation.*

Proof. From Theorem 2.4 we get the existence of a zero measure set $\mathscr{M}$ such that $Q \backslash \mathscr{M}$ is contained in the union of the characteristic curves coming from ∂Q_-. Letting χ denote the indicator function, the product function

$$(t, y) \in \mathbb{R}^{p+2} \to \chi_{[t^-(y), y_1]}(t) \chi_{Q \backslash \mathscr{M}}(y)$$

is measurable since the function $t^-(\cdot)$ is continuous. Since f is bounded in Q, the function $\chi_{[t^-(y), y_1]}(t) \chi_{Q \backslash \mathscr{M}}(y) f^2 (Y(t, y))$ is integrable in $[0, T] \times Q$. We conclude that u^2 is integrable in Q.

We note that the function $t \mapsto u(Y(t, y)) \in C^1(]t^-(y), t^+(y)[)$ and $\frac{\mathrm{d}}{\mathrm{d}s} u(Y(t, y)) = f(Y(t, y))$. The chain rule theorem provides with $t = y_1$:

$$(\beta|\nabla u) = f \text{ a.e.} \quad \text{in } Q.$$

We deduce

$$|u|_{1,\beta} = \|f\|_{L^2(Q)}.$$

Moreover from Lemma 2.2, we have $Y(t^-(y), y) \in \partial Q_-$ thus $u|_{\partial Q_-} = 0$. The estimate in the L^2-norm is a consequence of the equivalence of the semi-norm with the norm. □

Remark 3.4. We extend Definition 3.2 for a function $f \in L^2(Q)$ in a classical way. Let φ_n be a bounded by one function such that

$$\varphi_n(t, x) = \begin{cases} 1 & \forall (t, x) \in Q, \ \operatorname{dist}((t, x), \partial Q) \geqslant \dfrac{1}{n}, \\ 0 & \forall (t, x) \in \complement Q. \end{cases}$$

Now the product $\varphi_n f$ is regularized by convolution with a mollifying kernel and thus we get a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ converging towards f in $L^2(Q)$ (see [7] for example). The sequence $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, since formula (9) provides a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ verifying $(\beta|\nabla u_n) = f_n$ in $L^2(Q)$. Thus we have the convergence of $\{u_n\}_{p\in\mathbb{N}}$ towards u in the $H_0(\beta, Q)$-norm.

Conversely, for every $y \in Q \backslash \mathscr{M}$, the incoming time $t^-(y)$ can be defined and (9) is meaningful, which with estimates (10), provide a solution belonging to $H_0(\beta, Q)$.

3.1. The tangency condition and Nagumo's theorem

Now we introduce some well-known definitions of variational geometry (see for example [17]). For K a subset of X a Hilbert space, the distance function to K is defined by

$$d_K(x) = \inf_{z\in K} (\|x - z\|_X) \text{ if } K \neq \emptyset \quad \text{and} \quad d_K(x) = \infty \text{ otherwise.}$$

Definition 3.5. Let $K \subset X$ be a non-empty subset and let $\overline{x} \in K$ be given. Then $w \in X$ is tangent to K at the point $\overline{x}$ if

$$\lim_{h \to 0^+} \inf \frac{1}{h} d_K(\overline{x} + hw) = 0.$$

If $\overline{x} \in X$, then $w \in X$ is tangent to K at the point $\overline{x}$ if

$$\lim_{h \to 0^+} \inf \frac{1}{h}(d_K(\overline{x} + hw) - d_K(\overline{x})) \leqslant 0.$$

The set of tangent vectors to K at the point $\overline{x}$ is called the contingent cone or the Bouligand cone and is denoted by $T_K(\overline{x})$.

For $m < p+2$ let $h: \mathbb{R}^{p+2} \to \mathbb{R}^m$ be a C^1 function. Introduce a closed subset $D \subset \mathbb{R}^m$, and define $K \subset \mathbb{R}^{p+2}$ by $K = \tilde{C}_y \times \mathbb{R} \cap h^{-1}(D)$. The constraint set is such that it has to belong to a characteristic curve plus to the pre-image of D. Then we have (see [17, p. 221, 15]).

Lemma 3.6. *Assume $h \in C^1(\mathbb{R}^{p+2}; \mathbb{R}^m)$ to be of maximal rank on K (i.e., the rank of $Dh(z)$ equals m for all $z \in K$) and that the following transversality condition is satisfied*:

$$Dh(\overline{x})[T_{\tilde{C}_{\overline{x}_1} \times \mathbb{R}}(\overline{x})] - T_D(h(\overline{x})) = \mathbb{R}^m,$$

where $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^{p+1} \times \mathbb{R}$. Then we have

$$T_K(\overline{x}) = \{w \in \mathbb{R}^{p+2},\ w \in T_{\tilde{C}_{\overline{x}_1} \times \mathbb{R}}(\overline{x}) \text{ and } Dh(\overline{x})w \in T_D(h(\overline{x}))\}. \tag{11}$$

Please remark that since β is regular, then $T_{\tilde{C}_{\overline{x}_1} \times \mathbb{R}}(\overline{x}) = \operatorname{span}\{\beta(\overline{x}_1)\} \times \mathbb{R}$. For what follows, the Nagumo theorem for differential equations is given (see [2, p. 29 and p. 354]).

Theorem 3.7 (*Nagumo's theorem*). *Let X be an Hilbert space, K a locally closed subset and let $g \in C^0(K; X)$ be bounded. Then for all $\rho_0 \in K$ there exists $\rho \in C^1([0, T]; X)$ verifying*:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = g(\rho(t)) & \forall\, t \in (0, T), \\ \rho(0) = \rho_0, & \\ \rho(t) \in K & \forall\, t \in [0, T] \end{cases} \tag{12}$$

if and only if

$$\forall \varphi \in K, \quad g(\varphi) \in T_K(\varphi). \tag{13}$$

Let us give the main result of this article. Let $K \subset \overline{Q} \times \mathbb{R} \cap h^{-1}(D)$ be a closed subset.

Theorem 3.8. *Assume $(y, 0) \in K, \forall y \in \partial Q_-$. Then a sufficient condition for existence and uniqueness of solutions $u \in H_0(\beta, Q)$ verifying*:

$$\begin{cases} (\beta|\nabla u) = f \quad \text{for } y \in Q, \\ (y, u(y)) \in K \,(\text{i.e., } h(y, u(y)) \in D,\ \text{a.e. } y \in Q) \end{cases} \tag{14}$$

is given by for $F = (\beta, f)^t$ then

$$\begin{aligned} &\forall\,(y, r) \in K, \quad Dh(y, r)[\operatorname{span}\{\beta(y)\} \times \mathbb{R}] - T_D(h(y, r)) = \mathbb{R}^m, \\ &F(y, r) \in \operatorname{span}\{\beta(y)\} \times \mathbb{R}, \quad Dh(y, r)F(y, r) \in T_D(h(y, r)). \end{aligned} \tag{15}$$

Proof. Using Lemmas 3.1 and 3.3 the characteristic curves method reduces the $H_0(\beta, Q)$ solutions to the Transport equation subject to a constraint (9) to the solution H of the ordinary system of equations subject to a constraint (8).

Since $K \subset Q_1 \times \mathbb{R}$ with Q_1 bounded, Nagumo's theorem (12) applies and a necessary and sufficient condition (13) for existence and uniqueness of solutions to problem (8) is obtained. Condition (15) is nothing else than the tangency condition of Nagumo's theorem. $\square$

Remark 1. Condition (15) becomes necessary if the subset K is moreover constituted of points reachable with the Transport equation.

4. Algorithms and numerical experiments

The sufficient condition (15) obtained in the previous section is of limited practical use if it is not satisfied. If this condition is not satisfied, a simple remedy consists in modifying the right-hand side F in order to verify (15). Then the function f is changed, the little as possible, according to the obtained conditions. In what follows this strategy is exemplified with a subset K defined as follows. Let $D \subset \mathbb{R}$ be a closed subset, let $h: \ Q \times \mathbb{R} \to \mathbb{R}$ be a C^1 function, the rank of which is one, and the constraints subset K is defined by

$$K = \{(y, z) \in \overline{Q} \times \mathbb{R}, h(y, z) \in D\}. \tag{16}$$

Hereafter the domain Q and the function h (where y will be changed to (t, x)) are specified. Set $D = \mathbb{R}_-$, $\Omega = (0, 1)$, $Q = (0, T) \times \Omega$ and define:

$$h: \ \overline{Q} \times \mathbb{R} \to \mathbb{R},$$
$$((t, x), r) \mapsto \sin(10r) - \sin(x).$$

The derivative of h is $Dh(t, x, z) = (0, -\cos(x), 10\cos(10z))^t$. Since $\cos(x) \neq 0$ in Q, the rank of h is one. The vector valued function β is defined by $\beta(t, x) = (1, x + t)^t$, the function f is defined by $f(t, x) = x + (x + t)t$ for all $(x, t) \in Q$ and we set $F = (\beta, f)^t$. It is easy to check that the transversality condition in (15) is verified since $Dh(t, x, r)F(t, x, r) - T_D(h(t, x, r)) = \mathbb{R}$. Let sgn^- be defined by

$$\mathrm{sgn}^-(z) = \begin{cases} 1 & \text{if } z < 0, \\ 0 & \text{if } z \geqslant 0. \end{cases}$$

Now some technicalities concerning the contingent cone are given. Recall that the positive and negative parts of w are, respectively, denoted by $w^+ = \max(w, 0)$ and by $w^- = -\max(-w, 0)$. We have:

Lemma 4.1. *For all z the contingent cone $T_D(z)$ is defined by*

$$T_D(z) = \{w^+ \mathrm{sgn}^-(z) - w^- \ \ \forall w \in \mathbb{R}\}.$$

It is a convex subspace, so there exists a projector operator on $T_D(z)$ which is denoted by $\Pi_{T_D(z)}$ and given by

$$\Pi_{T_D(z)}\theta = \mathrm{sgn}^-(z)\theta^+ - \theta^-. \tag{17}$$

The contingent cone $T_D(z)$ is characterized with $\theta \in T_D(z)$ if and only if $\theta = \Pi_{T_D(z)}\theta$.

Gathering the previous results and the sufficient condition (15) we have:

Lemma 4.2. *Let β, f and h be given as previously. Set $F = (\beta, f)^t$, $K = \overline{Q} \times \mathbb{R} \cap h^{-1}(D)$, then $\forall \psi \in K$ the contingent cone $T_K(\psi)$ is given by*

$$T_K(\psi) = \{w \in \mathrm{span}[\beta \times \mathbb{R}] \text{ such that } Dh(\psi)w \in T_D(h(\psi))\}. \tag{18}$$

By using (17) and the characterization of T_D, condition (15) $F(t, x) \in T_K((t, x, u))$ is specified as

$$Dh(t, x, u)F(t, x) = \mathrm{sgn}^-(h(t, x, u))[Dh(t, x, u)F(t, x)]^+ - [Dh(t, x, u)F(t, x)]^-. \tag{19}$$

The Transport equation is a linear equation but due to the constraint, the problem becomes a nonlinear problem because of the nonlinear equation (19) to be solved. The resolution of Eq. (19) requires the modification of F in one case: if $h(\cdot, u) \geqslant 0$ and if $Dh(\cdot, u)F(\cdot) > 0$, then it is sufficient to impose $Dh(\cdot, u)F(\cdot) = 0$.

For solving problem (14), the following algorithm is proposed:

- assume u_k to be known
  (i) if $h(t,x,u_k)\leqslant 0$, $\widetilde{f}(u_k)=f$, otherwise compute $\widetilde{f}(u_k)$ verifying

$$Dh((\cdot,u_k)(\beta,\widetilde{f}(u_k))^t=\operatorname{sgn}^-(h(\cdot,u_k)[Dh(\cdot,u_k)(\beta,\widetilde{f}(u_k))^t]^+-[Dh(\cdot,u_k)(\beta,\widetilde{f}(u_k))^t]^- \tag{20}$$

  (ii) compute u_{k+1} solution to

$$\begin{cases}(\beta|\nabla u_{k+1})=\widetilde{f}(u_k) & \text{for } (t,x)\in Q,\\ u_{k+1}(t,x)=0 & \text{for } (t,x)\in\partial Q_-\end{cases} \tag{21}$$

- if $\|u_{k+1}-u_k\|_{H_0(\beta,Q)}\leqslant\varepsilon$ stop; else $k=k+1$ go to (i).

Now let us make it clear how to compute $\widetilde{f}$. Condition (20) reads: if $h(\cdot,u_k)\geqslant 0$ and if $Dh(\cdot,u_k)F(\cdot)>0$ then $Dh(\cdot,u)F(\cdot)=0$ so we have $\widetilde{f}(u_k)=-D_1h(\cdot,u_k)\beta/D_2h(\cdot,u_k)$; $\widetilde{f}(u_k)=f$ otherwise. Let us rewrite that in a functional way for proving the convergence of the algorithm.

$$\begin{aligned}\widetilde{f}(u_k)&=\operatorname{sgn}^-(h(\cdot,u_k))f+(1-\operatorname{sgn}^-(h(\cdot,u_k)))\\&\quad\times\left[\operatorname{sgn}^+(Dh(\cdot,u_k)F)-\frac{D_1h(\cdot,u_k)\beta}{D_2h(\cdot,u_k)}+(1-\operatorname{sgn}^+(Dh(\cdot,u_k)F))f\right],\\ \widetilde{f}(u_k)&=f-(1-\operatorname{sgn}^-(h(\cdot,u_k)))\frac{\operatorname{sgn}^+(Dh(\cdot,u_k)F)}{D_2h(\cdot,u_k)}(Dh(\cdot,u_k)F).\end{aligned} \tag{22}$$

Define

$$\operatorname{sign}^+(z)=\begin{cases}1 & \text{if } 0\leqslant z\\ 0 & \text{if } z<0,\end{cases}$$

set $G(u_k)=(Dh(\cdot,u_k)F)^+/D_2h(\cdot,u_k)$ and finally we get the following expression for the function $\widetilde{f}$:

$$\widetilde{f}(u_k)=f-\operatorname{sign}^+(h(\cdot,u_k))G(u_k). \tag{23}$$

Since we want Lemma 4.2 to apply, we need a continuous right-hand side for the Transport equation. So, we will deal with sign_n^+

$$\operatorname{sgn}_n^+(t)=\begin{cases}0, & t\leqslant-\dfrac{1}{n},\\ n\left(t+\dfrac{1}{n}\right), & -\dfrac{1}{n}\leqslant t\leqslant 0,\\ 1, & 0\leqslant t\end{cases} \tag{24}$$

a continuous perturbation of the sign^+ function, ensuring that the modified right-hand side belongs to the contingent cone (since the cut off function is changed only on the negative part). In what follows, for n large enough, we replace sign^+ by sign_n^+ in the expression of $\tilde{f}$ in (23).

Before giving a convergence result for the algorithm (20)–(21), let us recall that the Transport operator $T\in\mathscr{L}(H_0(\beta,Q),L^2(Q))$ is an isomorphism, and T^{-1} is monotone.

Lemma 4.3. *Assume the function G to be Lipschitz, non-decreasing and non-negative, the function h to be non-decreasing with respect to its second argument and the function f to be non-negative. Let $(\underline{u},\overline{u})=(T^{-1}\tilde{f}(T^{-1}f),T^{-1}f)$. Then starting the algorithm* (20)–(21) *with $u_0\in[\underline{u},\overline{u})$ provides a sequence $(u_n)_{n\in\mathbb{N}}\subset H_0(\beta,Q)$ converging towards $u\in H_0(\beta,Q)\cap K$ a solution to the problem*

$$Tu=f-\operatorname{sign}_n^+[h(\cdot,u)]G(u)=\tilde{f}(u). \tag{25}$$

Proof. We use a monotone iterative technique. First, remark that $(\underline{u}, \overline{u})$ define a couple of lower-upper solutions to (25). Since $\tilde{f}$ is non-increasing, we have

$$\left.\begin{array}{l} v \leqslant w, \\ Tv \leqslant \tilde{f}(w), \quad \tilde{f}(v) \leqslant Tw \end{array}\right\} \Longrightarrow Tv \leqslant \tilde{f}(w) \leqslant \tilde{f}(v) \leqslant Tw. \tag{26}$$

With $u_0 = \overline{u}$, $u_1 = T^{-1}\tilde{f}(\overline{u})$, define the sequence $u_n = T^{-1}\tilde{f}(u_{n-1})$ for $2 \leqslant n$. Since, G is non-negative, we have $\tilde{f}(\overline{u}) \leqslant f$, and $u_2 = T^{-1}\tilde{f}(u_0) \leqslant u_0 = T^{-1}f$ (T^{-1} is monotone). We deduce: $u_3 = T^{-1}\tilde{f}(u_2) \geqslant T^{-1}\tilde{f}(u_0) = u_1$. Let k be fixed and set $v_k = u_{2k+1}$; $w_k = u_{2k}$. We have $v_0 \leqslant v_1 \leqslant w_1 \leqslant w_0$. Arguing by induction we have

$$\left.\begin{array}{l} v_{k-1} \leqslant v_k \leqslant w_k \leqslant w_{k-1}, \\ Tv_{k+1} = \tilde{f}(w_k), \quad \tilde{f}(v_k) = Tw_{k+1} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} Tv_{k+1} = \tilde{f}(w_k) \leqslant \tilde{f}(v_k) = Tw_{k+1}, \\ Tw_{k+1} = \tilde{f}(v_k) \leqslant \tilde{f}(v_{k-1}) = Tw_k, \\ Tv_k \tilde{f}(w_{k-1}) \leqslant \tilde{f}(w_k) = Tv_{k+1}. \end{array}\right. \tag{27}$$

It is easy to prove that $(v_k)_{k\in\mathbb{N}}$ is a non-decreasing sequence and $(w_k)_{k\in\mathbb{N}}$ is a non-increasing sequence. To summarize (v_k, w_k) are sequences converging pointwise and in L^p toward (v, w) verifying, $v \leqslant w$, for $1 \leqslant p < \infty$. On the one hand, we know the function $\tilde{f}$ to be continuous and monotone, and the sequences (v_k, w_k) to be bounded and monotone, thus we have thanks to Lebesgue's dominated convergence theorem, the following convergence in L^p for $1 \leqslant p < \infty$

$$\lim_{n\to+\infty} \tilde{f}(v_n) = \tilde{f}(v), \quad \lim_{n\to+\infty} \tilde{f}(w_n) = \tilde{f}(w).$$

On the other hand, we have $v, w \in H_0(\beta, Q)$ which verify

$$v \leqslant w, \quad Tv = \tilde{f}(w), \quad Tw = \tilde{f}(v)$$

that is to say (v, w) is a pair of quasi-solutions to (25). Since, the function $\tilde{f}$ is Lipschitz, there exists a constant $0 < k$ such that

$$-k(w - v) \leqslant \tilde{f}(w) - \tilde{f}(v).$$

Gathering the previous inequality and the definition of a quasi-solution, we have

$$Tw - Tv - k(w - v) \leqslant 0. \tag{28}$$

Defining $V = \mathrm{e}^{-kt}v$; $W = \mathrm{e}^{-kt}w$, (28) leads to

$$TW - TV \leqslant 0$$

which implies $W \leqslant V$ and $w \leqslant v$ thus $v = w = u$ is a solution to (25). □

The hypothesis concerning the non-negativity of the function f is used for deriving a simple upper solution and can be weakened.

The section is ended with a numerical example. The numerical simulations have been realized with an implicit finite differences scheme with 150 points in time and 150 points in space. It is easy to check that the hypotheses of Lemma 4.3 are satisfied, thus the algorithm converges. The convergence of the iterative procedure is reached with at most 200 iterations. In Fig. 1 the solution without accounting for the constraint is depicted on the left ($u(t, x) = tx$), and on the right, the constraint function $(t, x) \mapsto h(t, x, u(t, x))$ is shown. In Fig. 2, the solution computed with the algorithm (20)–(21) is presented (left) and the residual error for the constraint (i.e., the positive part of $(t, x) \mapsto h(t, x, u(t, x))$) (right).

The residual error for the constraint is essentially due to the mesh size. If the mesh size decreases then the residual error decreases as is shown in the following table.

Convergence of the residual L^∞-error for the constraint

Number of points in each direction	200	400	800	1600
Residual L^∞-error	0.0094	0.0064	0.0033	0.00154

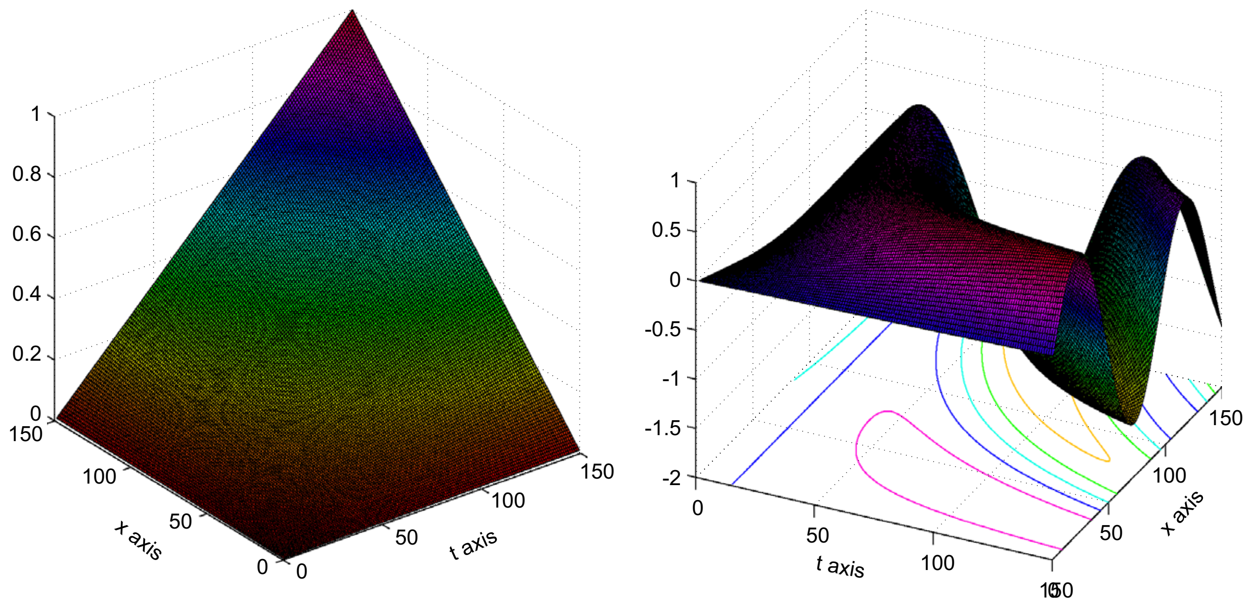

Fig. 1. Solution without constraint: (left), constraint for this solution (right).

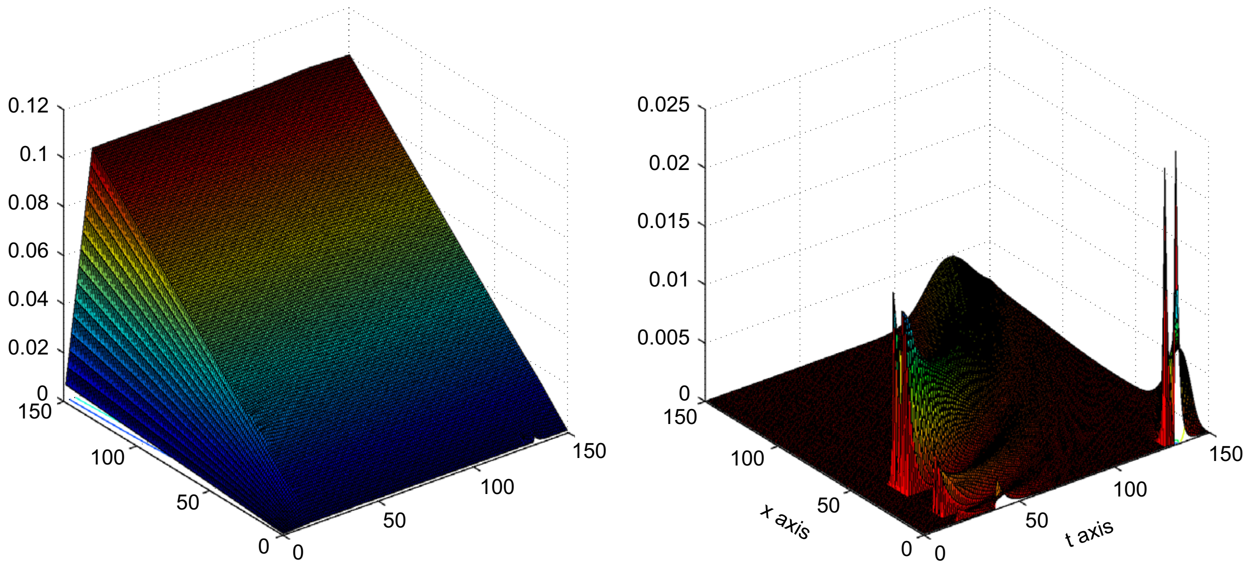

Fig. 2. Solution verifying the constraint (left), constraint residual error for this solution (right).

4.1. Conclusion

For the case of the Transport equation, Nagumo's theorem has been generalized, allowing this handling of a non-convex set of constraints. An efficient algorithm has been proposed when the set of constraints is the inverse range of a convex subset of $\mathbb{R}$ through a C^1 full rank application h. This algorithm allows the computation of the solution to problem (14) when the sufficient condition (15) is not verified and has been proved to be convergent. The equation obtained (25) can be interpreted as a generalized Langrange multiplier [17]. Numerical results have been presented illustrating the efficiency of this algorithm.

The method presented can be generalized to the case of a convex subset of $\mathbb{R}^p$ (see [15]) and will be presented in a forthcoming work.

References

[1] H. Amann, Gewohnliche Differentialgleichungen, second ed., de Gruyter, Berlin, 1995.
[2] T. Aubin, Viability Theory, Birkauser, Bale, 1995.
[3] C. Bardos, Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels: théorèmes d'approximation application à l'équations de transport, Ann. Sci. Ecole Norm. Sup. 4 série tome 3 (2) (1970) 185–233.
[4] R. Beals, V. Protopopescu, Abstract time-dependent transport equations, J. Math. Anal. Appl. 121 (1987) 370–405.
[5] O. Besson, J. Pousin, Hele–Shaw approximation for resin transfer molding, ZAMM 85 (4) (2005) 227–241.
[6] A. Bressan, Notes on the Boltzmann equation, Summer school at S.I.S.S.A. Trieste, 2005.
[7] H. Brézis, Analyse fonctionnelle, Masson, Paris, 1985.
[8] P. Erdi, J. Toth, Mathematical Models of Chemical Reactions, Manchester University Press, Manchester, 1989.
[9] L.C. Evans, R. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
[10] L. Hormander, Lectures on Nonlinear Hyperbolic Differential Equations, Springer, Berlin, 1997.
[11] S. Keeling, W. Ring, Medical image registration and interpolation by optical flow with maximal rigidity, J. Math. Imaging and Vision (2005) 47–65.
[12] R.H. Martins Jr., Nonlinear Operators and Differential Equations in Banach Spaces, Wiley-Interscience, New York, 1976.
[13] J. Necas, Les méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.
[14] M. Picq, Formulation temps-espace pour l'équation du Transport. ⟨http://maths.insa-lyon.fr/∼picq/25septblaise09.pdf ⟩.
[15] M. Picq, Résoluton de l'équation du Transport sous contraintes, Ph.D. Thesis Institut C. Jordan INSA de Lyon code BIU 50/210/9, 2007.
[16] J. Pousin, P. Pebay, O. Diallo, M. Picq, Solving the Transport Equation Subject to an Affine Constraint, European Congress on Computational Methods in Applied Sciences and Engineering, ECCOMAS, 2004.
[17] R.T. Rokafellar, J.B. Wets, Variational Analysis, Springer, Berlin, 2005.